Plan

- Why Quantum Mechanics?
- Postulates:
  - Postulate 1: The state of a system
  - Postulate 2: Composite systems
  - Postulate 3: Observable quantities
  - Postulate 4: Measurement
  - Postulate 5: Evolution of the state

Postulate 2: The state space of a composite physical system is given by the tensor product of the state space of its component.

A new Hilbert space is formed combining the Hilbert spaces $\mathcal{H}_a$ and $\mathcal{H}_b$ through a tensor product.

The new Hilbert space has dimension $h_a \cdot h_b$.

A set of basis states can be obtained from the basis state of each subsystem. For the two qubits for example we get

$$|\Psi_a\rangle|\Psi_b\rangle = |\Psi_a\rangle \otimes |\Psi_b\rangle = \begin{pmatrix} \alpha_a \\ \beta_a \end{pmatrix} \otimes \begin{pmatrix} \alpha_b \\ \beta_b \end{pmatrix} = \begin{pmatrix} \alpha_a \begin{pmatrix} \alpha_b \\ \beta_b \end{pmatrix} \\ \beta_a \begin{pmatrix} \alpha_b \\ \beta_b \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_a\alpha_b \\ \alpha_a\beta_b \\ \beta_a\alpha_b \\ \beta_a\beta_b \end{pmatrix}$$

Note: There are states which cannot be written as a product of state of each system, e.g.

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq |\Psi_a\rangle|\Psi_b\rangle$$

Composition of operators are done in a similar way the tensor product of operators for two qubits (each being a 2×2 matrix) is obtained by

$$\mathcal{O}^1 \otimes \mathcal{O}^2 = \begin{pmatrix} o^1_{00} & o^1_{01} \\ o^1_{10} & o^1_{11} \end{pmatrix} \otimes \begin{pmatrix} o^2_{00} & o^2_{01} \\ o^2_{10} & o^2_{11} \end{pmatrix} = \begin{pmatrix} o^1_{00} \begin{pmatrix} o^2_{00} & o^2_{01} \\ o^2_{10} & o^2_{11} \end{pmatrix} & o^1_{01} \begin{pmatrix} o^2_{00} & o^2_{01} \\ o^2_{10} & o^2_{11} \end{pmatrix} \\ o^1_{10} \begin{pmatrix} o^2_{00} & o^2_{01} \\ o^2_{10} & o^2_{11} \end{pmatrix} & o^1_{11} \begin{pmatrix} o^2_{00} & o^2_{01} \\ o^2_{10} & o^2_{11} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} o^1_{00}o^2_{00} & o^1_{00}o^2_{01} & o^1_{01}o^2_{00} & o^1_{01}o^2_{01} \\ o^1_{00}o^2_{10} & o^1_{00}o^2_{11} & o^1_{01}o^2_{10} & o^1_{01}o^2_{11} \\ o^1_{10}o^2_{00} & o^1_{10}o^2_{01} & o^1_{11}o^2_{00} & o^1_{11}o^2_{01} \\ o^1_{10}o^2_{10} & o^1_{10}o^2_{11} & o^1_{11}o^2_{10} & o^1_{11}o^2_{11} \end{pmatrix}$$

There is also an operator called the partial trace where the trace is taken over a subset of the state such as the state of a second system. E.g. suppose we have two system in a state of the form $\Psi_{Bell} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. If we take the partial trace over the second system we get

$$\rho_1 = Tr_2[\rho] = \sum_i \langle i_2|\rho|i_2\rangle = \frac{1}{2} Tr_2 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Postulate 3: An observable quantity $\mathcal{O}$ is represented by an Hermitian operator acting on this Hilbert space.

Observables are Hermitian operators. By the spectral decomposition theorem we can write these operators as:

$$\mathcal{O} = \sum_i o_i P_{o_i} \tag{1}$$

where $P_{o_i} = |\Psi_{o_i}\rangle\langle\Psi_{o_i}|$ and $|\Psi_i\rangle$ are the eigenstates of $\mathcal{O}$. (Let's assume that the eigenvalues are non-degenerate and thus P_{o_i} are orthogonal, $P_{o_i}P_{o_j} = \delta_{ij}P_{o_i}$ and $P_{o_i}^\dagger = P_{o_i}$.

The expectation (average value) of an operator $\mathcal{O}$ is given by

$$\langle\mathcal{O}\rangle = \langle\Psi|\mathcal{O}|\Psi\rangle \tag{2}$$

or if we are not in a pure state

$$\langle\mathcal{O}\rangle = Tr[\mathcal{O}\rho] = \sum_i \lambda_i\langle\Psi_i|\mathcal{O}|\Psi_i\rangle \tag{3}$$

(where $\rho = \sum_i \lambda_i|\Psi_i\rangle\langle\Psi_i|$).

Postulate 4: The action of measuring an observable is to project the state into an eigenstate of the observable $\mathcal{O}$.

The results of a measurement is given by the eigenvalues of the operator $\mathcal{O}$.

The probability to observe o_i when the state is $|\Psi\rangle$ is

$$p_{o_i} = |\langle\Psi|P_{o_i}|\Psi\rangle|^2$$

After the measurement the state is given by

$$|\Psi_{o_i}\rangle = \frac{P_{o_i}|\Psi\rangle}{\sqrt{\langle\Psi|P_{o_i}|\Psi\rangle}}$$

or in terms of the density matrix

$$\rho_f = |\Psi_{o_i}\rangle\langle\Psi_{o_i}| = \frac{P_{o_i}\rho P_{o_i}^{\dagger}}{Tr[P_{o_i}\rho P_{o_i}^{\dagger}]}$$

There is a generalizations of the concept of measurement to POVM (positive operator-valued measure) where the different measurements are not orthogonal (in operator space, i.e. they are represented by operators $M,\ N$ such that $Tr[MN] \neq 0$.

Postulate 5: The evolution of a system is given by the Schrödinger equation

$$-i\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle \tag{4}$$

where H is an operator called the Hamiltonian which defines the theory that we are working with (electromagnetism, QCD, gravity, string theory $\ldots$).

This equation is linear in $|\Psi\rangle$, i.e. if $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are two solution of the Schrödinger equation, $|\Psi_1\rangle + |\Psi_2\rangle$ is also a solution.

There is a formal solution for this equation

$$|\Psi(t)\rangle = e^{-i\int dtH}|\Psi(0)\rangle \tag{5}$$

If H is hermitian, $e^{-i\int dtH}$ is a unitary operator that we will call U. In quantum computation, U is a represent the algorithm. If we evolve the system for a time t_1 leading to U_1 and then for a second time t_2 leading to U_2, the total evolution can be obtained by combining the two evolution such that

$$U = U_2 U_1 \tag{6}$$

An algorithm can be build from smaller blocks, such as one and two bit operations, i.e. operations which acts on single qubits

$$U_1 = \mathbb{1} \otimes \mathbb{1} \dots U^k \dots \mathbb{1} \otimes \mathbb{1} \tag{7}$$

and

$$U_2 = \mathbb{1} \otimes \mathbb{1} \dots U^{k,l} \dots \mathbb{1} \otimes \mathbb{1} \tag{8}$$

If we are using the density matrix, we can see that it evolves as

$$\rho(t) = U \rho U^\dagger \tag{9}$$

as

$\rho = \sum_i \lambda_i |\Psi_i\rangle\langle\Psi_i|$,

$|\Psi_i\rangle \rightarrow U|\Psi_i\rangle$

and

$\langle\Psi_i| \rightarrow \langle\Psi_i|U^\dagger$

Notes:

1) There are different representations called the Scrödinger and Heisenberg representations. We have given above the Schrödinger one where state evolve and the operator are constant in time. But as the trace formula shows below

$$\begin{aligned}\langle \mathcal{O}(t) \rangle &= Tr[\mathcal{O}\rho(t)] \\ &= Tr[\mathcal{O}U\rho(t_o)U^\dagger] \\ &= Tr[U^\dagger \mathcal{O} U \rho(t_o)] \\ &= Tr[\mathcal{O}(t)\rho(t_o)]\end{aligned}$$

2) Note that there are two different “types” of evolution if we follow these postulates, the first one when a measurement is made and we have a projection of the state and the second one when the system evolved without outside perturbations when it follows the Schrödinger equation.

3) Although these postulates seems unsatisfactory in principle (why two types of evolution? why unitary evolution? why linear evolution?), they are incredibly good at describing experiments of small isolated systems.