Plan

- Why Quantum Mechanics?
- Postulates:
  - Postulate 1: The state of a system: Hilbert space
  - Postulate 2: Composite systems: Tensor product
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Why Quantum Mechanics?

- 1. Equilibrium of radiation with the walls of a cavity
- 2. Photoelectric effect
- 3. Discrete spectrum of radiation from atoms
- 4. Stability of atoms

Postulate 1: The state of a system is described by a vector in a Hilbert space $\mathcal{H}$.

A Hilbert space is complex vector space with a scalar product.

1. A vector space **V**is a space with the following properties:
   - Closure: Adding two vector will give a vector **v** + **u** $\in$ **V**
   - There is a **0** vector
   - Scalar multiplication: c**v** is also a vector
   - Existence of an inverse $\forall$ **v**, $\exists$ **-** v $\in$ V
   - Associativity **(**v+u)+ t = v+ (u+t)

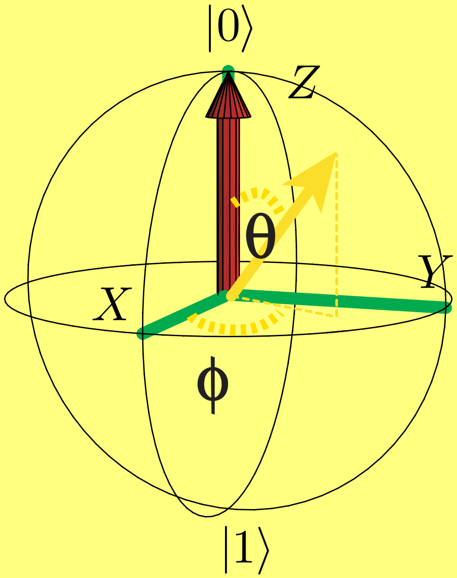
2. A scalar product is a function of (**v**,**u**)$\rightarrow$ c. We will write the scalar product of the se two states in terms of the Hermitian conjugate of one of them in the form: $\langle \mathbf{u} || \mathbf{v} \rangle$.

We describe the state as a vector with complex coefficient, e.g. for a qubit (2 state system)

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$\alpha,\ \beta$ are complex numbers, i.e. $= r_1 + ir_2$ for $r_i \in \Re$ and are called the *amplitude* to be in the respective states. When $\alpha \neq 0 \text{ or } 1$ we say that the state is in a superposition.

For 1 qubit the state can be written as $\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$ and has the following geometric interpretation:



This description of the state is called the Bloch sphere. (Only the relative phase between the amplitudes of the state can be observed).

The hermitian conjugate vector to $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ will be denoted by

$$\langle\Psi| = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}$$

($*$ means $r_1 + ir_2 \rightarrow r_1 - ir_2)$)

There scalar product will be written as

$$\langle\Phi||\Psi\rangle = \alpha^*_\phi\alpha_\psi + \beta^*_\phi\beta_\psi$$

Hilbert spaces require that the states are normalized so we will restrict the coefficients α and β such that $\langle\Psi||\Psi\rangle = \alpha\alpha^* + \beta\beta^* = 1$.

- OPERATORS

We will have operators which acts on this space: the evolution operator and the observables that we mention below. They are mathematical entities which associate one vector **v** $\in$ **V** to another **u** $\in$ **V**. In quantum mechanics we focus on linear operator and they can be represented by $n \times n$ matrices (when we are dealing with a Hilbert space of dimension n).

We will especially be interested in linear operators $\mathcal{O}$. By linear we mean that when they act on the state $|\Psi\rangle = \lambda_1 |\Psi_1\rangle + \lambda_2 |\Psi_2\rangle$ we get

$$\mathcal{O}|\Psi\rangle = \mathcal{O}(\lambda_1|\Psi_1\rangle + \lambda_2|\Psi_2\rangle) = \mathcal{O}\lambda_1|\Psi_1\rangle + \mathcal{O}\lambda_2|\Psi_2\rangle \quad (1)$$

E.g. for our qubits (2-d Hilbert space), an operator $\mathcal{O}$ will be a 2×2 matrices of the form:

$$\mathcal{O} = \begin{array}{c c} & \begin{array}{cc} \langle 0| & \langle 1| \end{array} \\ \begin{array}{c} |0\rangle \\ |1\rangle \end{array} & \begin{pmatrix} o_{00} & o_{01} \\ o_{10} & o_{11} \end{pmatrix} \end{array} \tag{2}$$

A basis set of operator are the Pauli matrices

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Any operator $\mathcal{O}$ can be written as a combination of the matrices above. Hermitian operators have real coefficients.

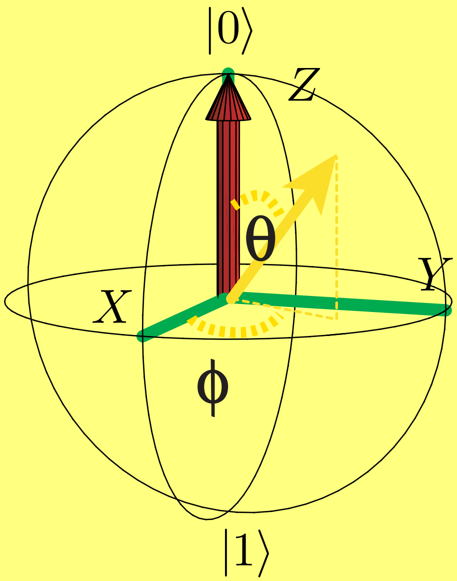
We can turn the state into an observable through the outer product, i.e.

$$|\Psi\rangle \to |\Psi\rangle\langle\Psi| \tag{3}$$

This is an Hermitian operator projecting in a 1 dimensional space. For $\Psi = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$, the density matrix looks like

$$\begin{aligned} \rho &= \frac{1}{2}\mathbb{1} + \alpha X + \beta Y + \gamma Z & (4)\\ &= \frac{1}{2}\mathbb{1} + \sin\theta\cos\phi X + \sin\theta\sin\phi Y + \cos\theta Z & (5) \end{aligned}$$

(Note that $\alpha^2 + \beta^2 + \gamma^2 = (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2 = 1$)



So does the inside of the sphere corresponds to the state of any physical system?

Yes, if ρ describe an incoherent superposition of states

$$\rho = \sum_i \lambda_i |\Psi_i\rangle\langle\Psi_i| \tag{6}$$

then the density matrix takes the above form but with $\alpha^2 + \beta^2 + \gamma^2 < 1$, and thus the whole state space of one qubit is S^2 ball.

Example of where this uncertainty can come from:

- 1) just preparation by an external system
- 2) by quantum correlation

The trace operator Tr is defined as

$$Tr[\mathcal{O}] = \sum_i \mathcal{O}_{ii} = \sum_i \langle \Psi_i | \mathcal{O} | \Psi_i \rangle \tag{7}$$

where $|\Psi_i\rangle$ is a complete and orthonormal set of state (i.e span the whole space $\mathcal{H}$). This sum can be thought of as the expectation of the operator $\mathcal{O}$.

The density matrix ρ has the property that

- 1. $Tr[\rho] = 1$
- 2. Positivity: $\forall\ |\Psi\rangle$, $\langle \Psi | \rho | \Psi \rangle \geq 0$.

It can be shown that the eigenvalues of $\rho \in \ [0, 1]$ and they are interpreted as the probability to be in the corresponding eigenvector.

A density matrix such that $Tr[\rho^2] = 1$ is said to be pure, otherwise it is mixed.

(The von Neuman entropy $\mathcal{S}$ is defined as $Tr[\rho\log\rho]$, it is a measure of how much information information a state has.)

There are many ways to prepare a given mixed state. E.g. we can prepare the state $\rho = \frac{1}{2}\mathbb{1}$ with an equal mixture of the state $|0\rangle, |1\rangle$ or an equal mixture of $\frac{|0\rangle\pm|1\rangle}{\sqrt{2}}$. There are no measurement that can distinguish these two preparations.

The state ρ contains all the information that can be accessed about this system.